Circles and Lines

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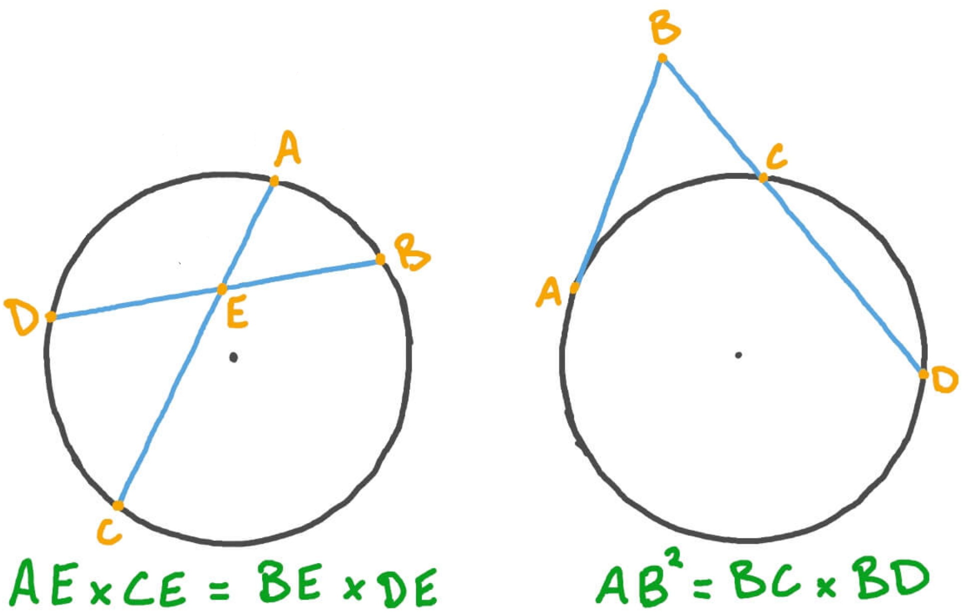
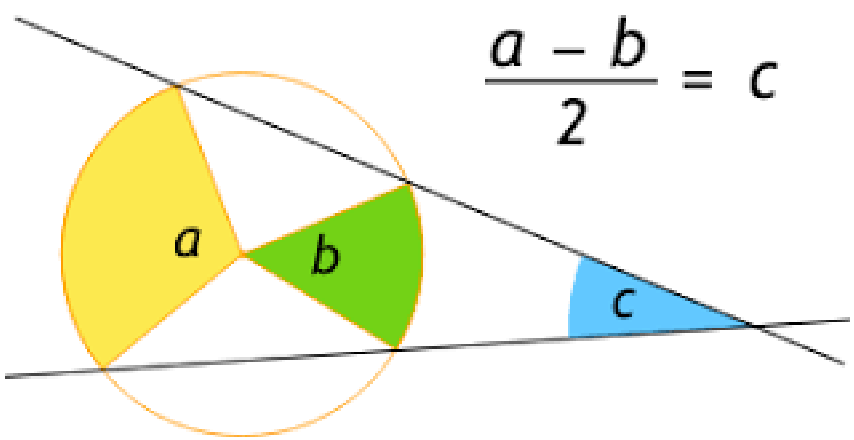
1 Concepts

Power of Points



(1.1)

Two Intersecting Secants Theorem

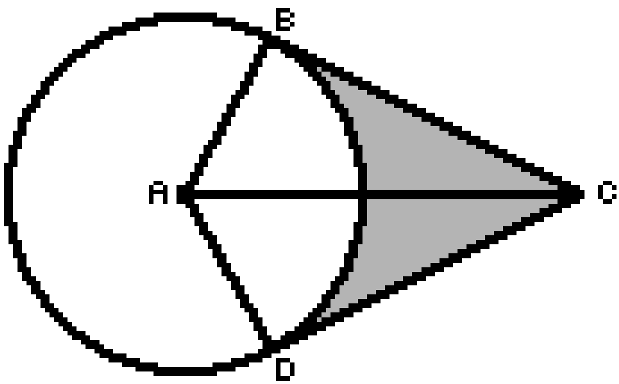


(1.2)

2 Practice Problems

2.1 Problem 1

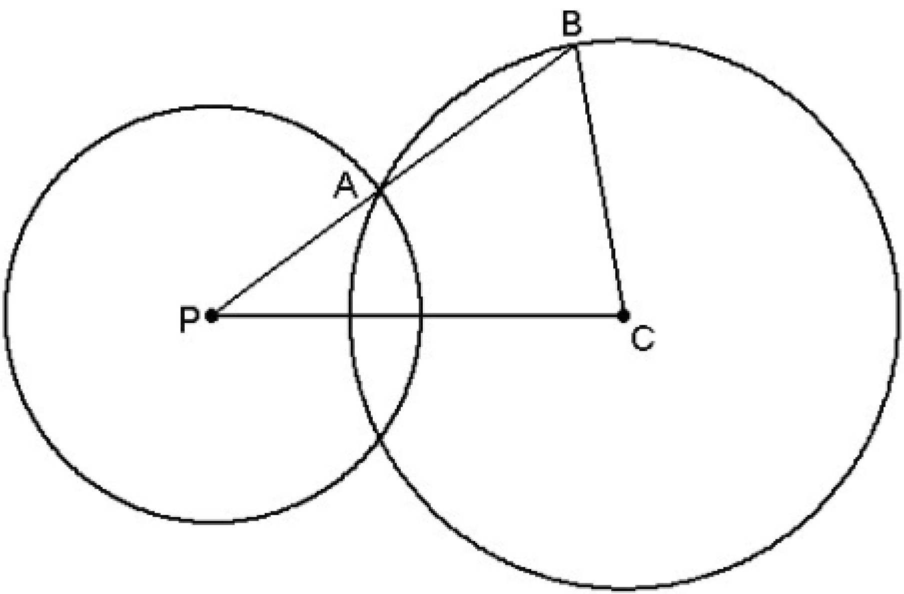
A circle of radius 3 units is drawn. Lines BC and DC are tangent to the circle. $\angle BCD$ is 60 degrees. Find the area of the shaded area.



(2.1)

2.2 Problem 2 (Brilliant)

In the figure below, the small circle with center P has a radius 3 and the large circle with center C has radius 4. PC has length 6 units. What is the length of AB?

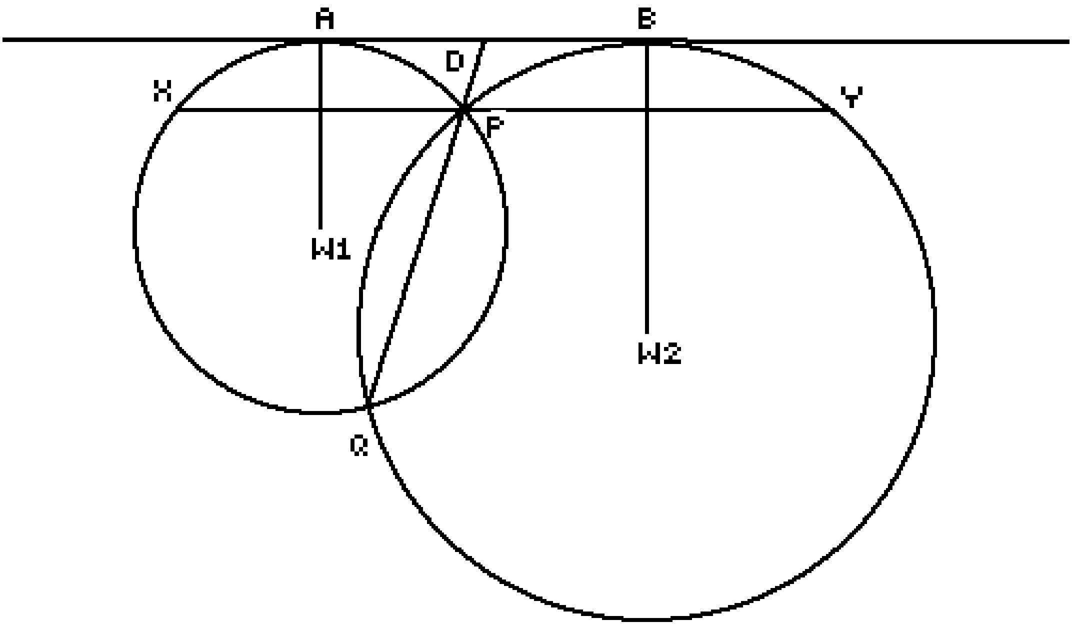


(2.2)

2.3 Problem 3 (AIME)

Circles ω_1 and ω_2 intersect at two points P and Q, and their common tangent line closer to P intersects ω_1 and ω_2 at points A and B, respectively. The line parallel to AB that passes through P intersects ω_1 and ω_2 for the second time at points X and Y, respectively. Suppose $PX = 10$, $PY = 14$, and $PQ = 5$. Then the area of trapezoid $XABY$ is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime.

Find $m + n$.



(2.3)

2.4 Problem 4 (AHSME)

Two tangents are drawn to a circle from an exterior point A; they touch the circle at points B and C respectively. A third tangent intersects segment AB in P and AC in R, and touches the circle at Q. If $AB = 20$, what is the perimeter of $\triangle APR$?

3 Solutions

3.1 Solution 1

$\triangle ABC$ is a $30-60-90$ triangle, with AB being 3 units and BC being $3\sqrt{3}$. The area of $\triangle ABC$ is $\frac{9\sqrt{3}}{2}$, meaning quadrilateral $ABCD$ has area $9\sqrt{3}$.Sector BAD has an angle of 120 degrees, a third of the circle. The area of the sector is $\frac{1}{3}\cdot 3^2\cdot\pi = 3\pi$. Therefore the shaded area is $9\sqrt{3}-3\pi$.

3.2 Solution 2

Denote the intersection of the large circle and PC as X From power of points, we know:

$$PA\cdot PB = PX\cdot PC$$

The length of PX is simply $PC - XC = 6-4 = 2$. So we have:

$$3\cdot(3+AB) = 2\cdot 6 = 12$$

$$3+AB = 4$$

$$AB = 1$$

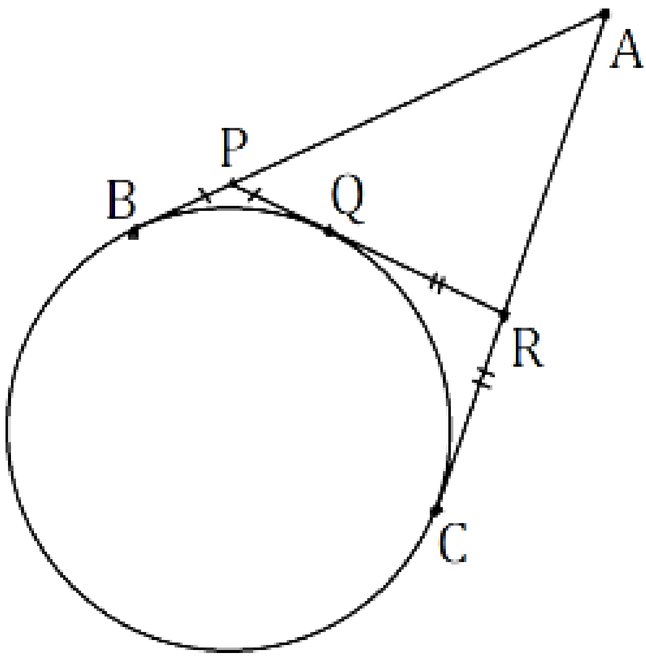
3.3 Solution 3

Let the intersection between AW_1 and XP be O_1, and BW_2 and PY be 0_2. Note the lines connecting A and W_1 are perpendicular to AB, as is BW_2. Because AB and XY are parallel, XP is perpendicular AW_1, meaning $A0_1 = 0_1P$, and the same applies to PO_2 and O_2Y. That means $12 = 0_10_2 = AB = 7+5 = 12$. Power of points states that $AD^2 = DB^2 = DP\cdot(DP+PQ)$, meaning $AD = DB = 6$. We have

$$36 = DP\cdot(DP+5)$$

, and solving we get $DP = 4$. To find the height of the quadrilateral, we use the Pythagorean theorem. One leg would be the height, the other would have length $DA - 0_1P = 1$ with a hypotenuse $DP = 4$. Solving for the height $\sqrt{4^2-1} = \sqrt{15}$. Using the formula for area of a quadrilateral, the area of $XYAB$ is $18\sqrt{15}$, so the answer is 33.

3.4 Solution 4



(3.1)

Image Credit: AOPS We are trying to find $AP + PR + AR$. $PR = PQ + QR = PB + RC$, which means $AP + PR + AR = AB + AC$, so the perimeter of the triangle is 40 units.