Trigonometry*

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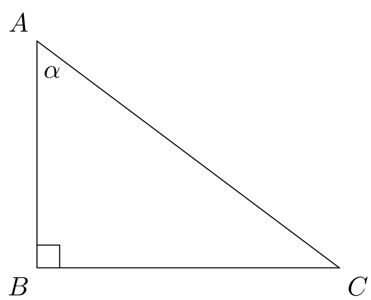
March 27, 2020

1 Introduction

Recall that if a triangle ABC is a right triangle with $\angle ABC = 90^\circ$ and $\angle A = \alpha$, then

$$\sin\alpha = \frac{BC}{AC}, \quad \cos\alpha = \frac{AB}{AC}, \quad \tan\alpha = \frac{BC}{AB}.$$



Definition (Angle of Elevation) The angle of elevation is the angle above the horizontal at which a viewer must look to see an object that is higher than the viewer.

Definition (Angle of Depression) The angle of depression is the angle below the horizontal at which a viewer must look to see an object that is below the viewer.

Formula (Area of a Triangle) If $a = BC$, $b = AC$, and $[ABC]$ is the area of $\triangle ABC$, then

$$[ABC] = \frac{1}{2}ab\sin c.$$

Formula (Law of Cosines) Let $a = BC$, $b = AC$, and $c = AB$ in $\triangle ABC$. The Law of Cosines states that

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Formula (Extended Law of Sines) If $a = BC$, $b = AC$, $c = AB$, and R is the radius of the circumcircle of $\triangle ABC$, then the Extended Law of Sines states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

*Many of the examples and content in the following handout have been adopted from Richard Rusczyk's *Introduction to Geometry*.

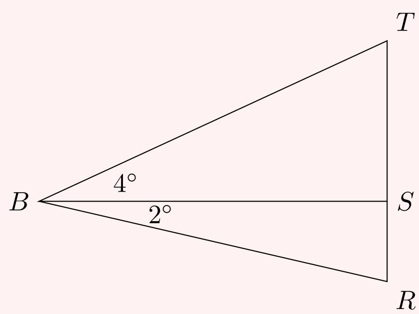
Formula (Sum and Difference of Angles)

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

2 Examples

Exercise 1. A bee is on a hill looking at a building. The building is 400 feet tall. The angle of elevation from the bee to the top of the building is $4°$ and the angle of depression from the bee to the bottom of the building is $2°$. What is the shortest distance the bee will have to fly to reach the building?

Solution 1. Denote by point B the position of the bee. Similarly, denote by points R and T the lowest and highest points on the building, and denote by point S the point on the building closest to the bee.



We are given that $RT = 400$ feet, so because $RT = RS + ST$, we can try to express each of RS and ST in terms of BS. Using $\triangle BST$ and $\triangle BSR$, we can write

$$\tan \angle TBS = \frac{ST}{BS}$$
$$\implies ST = BS \tan 4° \approx 0.070BS$$

and

$$\tan \angle RBS = \frac{RS}{BS}$$
$$\implies RS = BS \tan 2° \approx 0.035BS.$$

Finally, we have

$$\begin{aligned} RT &= RS + ST \\ &= 0.035BS + 0.070BS \\ &= 0.105BS. \end{aligned}$$

Using the fact that $RT = 400$ feet, we find that $BS \approx 3800$ feet.

Exercise 2. If s is the semiperimeter and R is the circumradius of $\triangle ABC$, show that $\sin A + \sin B + \sin C = \frac{s}{R}$.

Solution 2. The Law of Sines states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

so we have

$$\begin{aligned}\sin A + \sin B + \sin C &= \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \\ &= \frac{a+b+c}{2R} \\ &= \frac{s}{R}\end{aligned}$$

as desired.

Exercise 3. Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$? (Source: AMC)

Solution 3. Square both of the given equations to obtain the following:

$$\begin{aligned}\sin^2 a + 2\sin a \sin b + \sin^2 b &= \frac{5}{3} \\ \cos^2 a + 2\cos a \cos b + \cos^2 b &= 1.\end{aligned}$$

Taking the sum of the two equations and rearranging gives us the following

$$\sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b + 2(\cos a \cos b + \sin a \sin b) = \frac{8}{3}.$$

Then, because we know that $\sin^2\theta + \cos^2\theta = 1$, we can simplify to get

$$2(\cos a \cos b + \sin a \sin b) = \frac{2}{3}.$$

The expression inside the parenthesis is equivalent to $\cos(a-b)$. Therefore, we can rewrite the equation and solve.

$$\begin{aligned}2\cos(a-b) &= \frac{2}{3} \\ \cos(a-b) &= \frac{1}{3}\end{aligned}$$

Exercise 4. $ABCD$ is a square and M and N area the midpoints of $\overline{BC}$ and $\overline{CD}$, respectively. What is $\sin\angle MAN$? (Source: AHSME)

Solution 4. In this solution, assume that each side of the square has a length of 1. In this square, we know that $\overline{AM}$ and $\overline{AN}$ both have length $\frac{\sqrt{5}}{2}$. With both of these side lengths known, we can find $\sin \angle MAN$ if we know $[MAN]$ by using the equation for area from Exercise 2. To find $[MAN]$, we simply subtract the area from the missing triangles.

$$\begin{aligned} [MAN] &= 1 - \frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{3}{8} \end{aligned}$$

Finally, we can apply the equation for the area of a triangle gives the:

$$\frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2 \sin \angle MAN = \frac{3}{8} \quad (1)$$

$$\implies \sin \angle MAN = \frac{3}{5}. \quad (2)$$

Exercise 5. A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$. (Source: AIME)

Solution 5. Let P and Q be the points on the creased line closest to point B and point C, respectively. Let $AP = a$, $PQ = x$, and $AQ = b$. Since AB had a length of 12 before the fold and because a length of a was folded over, we have that $BP = 12 - a$. By the same token, we have that $CQ = 12 - b$. Furthermore, note that

$$\angle ABP = \angle PAQ = \angle ACQ = 60^\circ$$

since the original triangle was equilateral. Using the Law of Cosines on $\triangle ABP$ yields

$$\begin{aligned} AP^2 &= BP^2 + AB^2 - 2(BP)(AB)\cos\angle ABP \\ a^2 &= (12-a)^2 + 9^2 - 2(12-a)(9)\cos 60 \\ a^2 &= 144 - 24a + a^2 + 81 - 108 + 9a \\ \implies a &= \frac{39}{5}. \end{aligned}$$

Similarly, applying the Law of Cosines on $\triangle ACQ$ gives

$$\begin{aligned} AQ^2 &= AC^2 + CQ^2 - 2(AC)(CQ)\cos\angle ACQ \\ b^2 &= 3^2 + (12-b)^2 - 2(3)(12-b)\cos 60 \\ b^2 &= 9 + 144 - 24b + b^2 - 36 + 3b \\ \implies b &= \frac{39}{7}. \end{aligned}$$

Finally, applying the Law of Cosines on $\triangle APQ$ yields

$$\begin{aligned} PQ^2 &= AP^2 + AQ^2 - 2(AP)(AQ)\cos\angle PAQ \\ x^2 &= \frac{39}{5}^2 + \frac{39}{7}^2 - 2\left(\frac{39}{5}\right)\left(\frac{39}{7}\right)\cos 60 \\ \implies x &= \frac{39\sqrt{39}}{35}. \end{aligned}$$

Thus, the answer is $39 + 39 + 35 = 113$.

3 Problem Set

Problem 1. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x+y)$? (Source: AIME)

Problem 2. Suppose that

$$\sin a + \cos b = \sqrt{5}.$$

Calculate

$$\frac{\tan a \sin b}{\tan b \sec a}.$$

Proposed by Deepu Pradeep.

Problem 3. In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$. (Source: AIME)

Problem 4. Equilateral $\triangle ABC$ has side length 2, M is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. What is the area of $\triangle CDM$? (Source: AMC)

Problem 5. Let $\triangle PQR$ be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers $a, b, c,$ and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$. (Source: AIME)